MATH20132 Calculus of Several Variables. 2020-21

Problems 4 Fréchet derivatives

*Fréchet differentiable **scalar-valued** functions.*

1. a. Prove, by verifying the definition, that the scalar-valued functions

i. $f : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto x(x+y)$ and

ii. $g : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto y(x-y)$

are Fréchet differentiable at a general point $\mathbf{a} = (\alpha, \beta)^T$ and find the Fréchet derivatives $df_{\mathbf{a}}$ and $dg_{\mathbf{a}}$.

b. Use your results to check your answers to Question 1 on Sheet 3.

2. i. Define the function $h : \mathbb{R}^3 \to \mathbb{R}$ by $h(\mathbf{x}) = xy + yz + xz$ where $\mathbf{x} = (x, y, z)^T$. Prove by verifying the definition that f is Fréchet differentiable at a general point $\mathbf{a} = (\alpha, \beta, \gamma)^T \in \mathbb{R}^3$ and find the Fréchet derivative $dh_{\mathbf{a}}$ of h at $\mathbf{a}$.

ii. Use your result to check your answer to Question 3 on Sheet 3.

*Fréchet differentiable **vector-valued** functions.*

3. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix},$$

for $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$. Prove, by *verifying the definition* that $\mathbf{f}$ is everywhere Fréchet differentiable and find the Fréchet derivative of $\mathbf{f}$ at a general point $\mathbf{a} = (\alpha, \beta)^T$.

4. i. Prove that the scalar-valued function $\mathbf{x} \mapsto x^2 y$ is everywhere Fréchet differentiable on $\mathbb{R}^2$. In lectures and Problems class this was done for the scalar-valued function $\mathbf{x} \mapsto xy^2$, simply copy that method for the second.

ii. Prove that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2 y \end{pmatrix},$$

is Fréchet differentiable at a general point $\mathbf{a} = (\alpha, \beta)^T \in \mathbb{R}^2$ and find the Fréchet derivative $d\mathbf{f}_{\mathbf{a}}$ of $\mathbf{f}$ at $\mathbf{a}$.

ii. Use the result from part i to check your answer to Question 7 on Sheet 3.

Note the difference in wording between this question and the previous one.

Jacobian Matrices and Gradient vectors

5. Write down the general Jacobian matrix in each of the following cases and then evaluate them at the given point.

i. $\mathbf{p}(r, \theta) = (r\cos\theta, r\sin\theta)^T$ at the point $(1, \pi)^T$,

ii. $g(u, v, w) = uv + 5u^2w$ at the point $(2, -3, 1)^T$,

iii. $\mathbf{r}(t) = (\cos t,\, \sin t,\, t)^T$, a helix in $\mathbb{R}^3$, at the point $t = 3\pi$.

6. By returning to Questions 1 and 2 find the gradient vectors of

i. $f : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto x(x+y)$ and

ii. $g : \mathbb{R}^2 \to \mathbb{R}, \mathbf{x} \longmapsto y(x-y)$

iii. $h : \mathbb{R}^3 \to \mathbb{R}$ by $\mathbf{x} \longmapsto xy + yz + xz$ where $\mathbf{x} = (x, y, z)^T$,

without using partial differentiation. Justify your argument.

7. By returning to Question 3 and 4 find the Jacobian matrices of $\mathbf{f}, \mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}, \text{ and } \mathbf{g}(\mathbf{x}) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$$

without using partial differentiation. Justify your argument.

Not *Fréchet differentiable*

8. *Recall the important result for* ***scalar-valued*** *functions*

$$\boxed{f \text{ Fréchet differentiable at } \mathbf{a} \implies \forall \text{ unit } \mathbf{v}, d_{\mathbf{v}}f(\mathbf{a}) \text{ exists and } d_{\mathbf{v}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \bullet \mathbf{v}.}$$

The contrapositive of this is the useful

$$\boxed{\begin{array}{c} \exists \text{ unit } \mathbf{v} : \text{either } d_{\mathbf{v}}f(\mathbf{a}) \text{ does not exist or } d_{\mathbf{v}}f(\mathbf{a}) \neq \nabla f(\mathbf{a}) \bullet \mathbf{v} \\ \implies f \text{ is } \textbf{not} \text{ Fréchet differentiable at } \mathbf{a} \end{array}} \qquad (1)$$

Define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(\mathbf{x})=\frac{x^2y}{x^2+y^2}\quad\text{if } \mathbf{x}\neq 0,\qquad f(\mathbf{0})=0.$$

Prove that f is **not** Fréchet differentiable at $\mathbf{0}$.

This function was seen in Question 11iii Sheet 1 where it was shown that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})=0$. *Since* $f(\mathbf{0})=0$ *this means* f *is continuous at* $\mathbf{0}$. *So we have an example illustrating the important*

$$\boxed{f \text{ continuous at } \mathbf{a} \;\not\Longrightarrow\; f \text{ is Fréchet differentiable at } \mathbf{a}.}$$

But f *was also seen in Question 14 on Sheet 3 where it was shown that the directional derivatives* $d_{\mathbf{v}}f(\mathbf{0})$ *exist for all unit* $\mathbf{v}$. *So we have an example of*

$$\boxed{\forall \text{ unit } \mathbf{v},\ d_{\mathbf{v}}f(\mathbf{a}) \text{ exists} \;\not\Longrightarrow\; f \text{ is Fréchet differentiable at } \mathbf{a}.}$$

In Question 8 we used that fact that there exists a unit $\mathbf{v}$ for which $d_{\mathbf{v}}f(\mathbf{0})\neq\nabla f(\mathbf{0})\bullet\mathbf{v}$ to deduce that f is not Fréchet differentiable at $\mathbf{0}$. In the following question we find an example of a function f for which $d_{\mathbf{v}}f(\mathbf{0})=\nabla f(\mathbf{0})\bullet\mathbf{v}$ for all unit $\mathbf{v}$ and yet f is still not Fréchet differentiable at $\mathbf{0}$.

9. (Tricky) Define the function $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(\mathbf{x})=\frac{xy^2\sqrt{x^2+y^2}}{x^2+y^4}\quad\text{if } \mathbf{x}\neq\mathbf{0};\ \ f(\mathbf{0})=0.$$

i. Prove that f is continuous at $\mathbf{0}$.

   Hint Note that $x\leq\sqrt{x^2+y^4}$, and similarly for y^2.

ii. Prove from first principles that the directional derivative exists in all directions, and further, satisfies $d_{\mathbf{v}}f(\mathbf{0})=\nabla f(\mathbf{0})\bullet\mathbf{v}$ for all unit vectors $\mathbf{v}\in\mathbb{R}^2$.

iii. Prove that, nevertheless, f is **not** Fréchet differentiable at $\mathbf{0}$.

This example illustrates two important points

$$\boxed{\text{continuous} \;\not\Longrightarrow\; \text{Fréchet differentiable.}}$$

and

$$\boxed{\forall \text{ unit } \mathbf{v}, d_{\mathbf{v}}f\,(\mathbf{a})\,\text{exists and } d_{\mathbf{v}}f(\mathbf{a})=\nabla f(\mathbf{a})\bullet\mathbf{v} \;\not\Longrightarrow\; \text{differentiable.}}$$

Additional Questions 4

10. *Product Rule for Gradient vectors* Assume for $f, g : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ that the gradient vectors $\nabla f(\mathbf{a})$ and $\nabla g(\mathbf{a})$ exist at $\mathbf{a} \in U$. Prove that $\nabla (fg)\,(\mathbf{a})$ exists and satisfies

$$\nabla (fg)\,(\mathbf{a}) = f(\mathbf{a})\,\nabla g(\mathbf{a}) + g(\mathbf{a})\,\nabla f(\mathbf{a})\,.$$

11. Prove that

$$f(\mathbf{x}) = \frac{xy}{\sqrt{x^2+y^2}},\ \mathbf{x} = (x,y)^T \neq \mathbf{0}, \quad f(\mathbf{0}) = 0,$$

is continuous but not Fréchet differentiable at $\mathbf{0}$.